Extremal Submodular Functions

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Modular and submodular functions

A function defined on a (lattice) of subsets of X is *modular* if

$$f(A)+f(B)=f(A\cup B)+f(A\cap B).$$

Example: a (signed) measure.
If $f(\emptyset)=0$ then it is *pointed*.
An outer measure has only

$$f(A)+f(B)\geq f(A\cup B)+f(A\cap B)$$

It corresponds to convexity and called *submodular*.

Generator set

Fix the *finite* base set X where $|X| = n$.

• Modular functions over X form an $n+1$-dimensional linear space.

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B).$$

• Conic (non-negative linear) combination of submodular functions is submodular.

• Sum of a modular and a submodular function is submodular.

• $\mathcal{G}$ is a generator if every submodular function is a conic combination of $\mathcal{G}$ up to a modular shift.

• The minimal generator set is unique up to scaling and a modular shift.

• Elements of this minimal generator set are *extremal*.

Extremal submodular functions

Problem Given $|X| = n$, find all extremal submodular functions.

The *dimension* is $d = 2^n - (n+1)$ after deducting the modular part.

The submodular cone has $2n!$ symmetries: reflection $A \mapsto X - A$.

It has m bounding inequalities.

Orbits: symmetrically equivalent extremal functions.

n	d	m	symm	# ext	orbits
3	4	6	12	5	3
4	11	24	48	37	7
5	26	80	240	117978	672
6	57	240	1440	$\approx 10^{28}$	$\approx 10^{25}$

Methods – 1

Theorem (Algebraic test)

The function f is extremal iff the rank of active constrains is $d-1$.
f_1 and f_2 are adjacent iff the rank of joint active constrains is $d-2$.

Adjacency decomposition

Take an extremal function and enumerate all adjacent extremal functions. — This is a $(d-1)$-dimensional problem with significantly smaller number of constraints on the average.

Can be applied iteratively. Minimal required resources are proportional to the total number of orbits.

Caveat: For $n=6$ there is an extremal function which has neighbors in more than 80% of all orbits.

Methods – 2

Double Description method

Start with d active constraints and d extremal functions for these constraints.
Add iteratively one new active constraint and update the set of extremal functions.

Requires bitmap operations on the adjacency matrix of constraints and functions.

The intermediate size can be exponentially larger than the final result. Hopefully not with a smart ordering of the constraints.

Experimental estimate

Running time of the $i+1$-st iteration $\approx$ (time of i-th iteration)$^{1.1}$.

Unfortunately we need 180 iterations.

Applied statistics.